Tutorial: Tensor Approximation in Visualization and Computer Graphics

Tensor Decomposition Models

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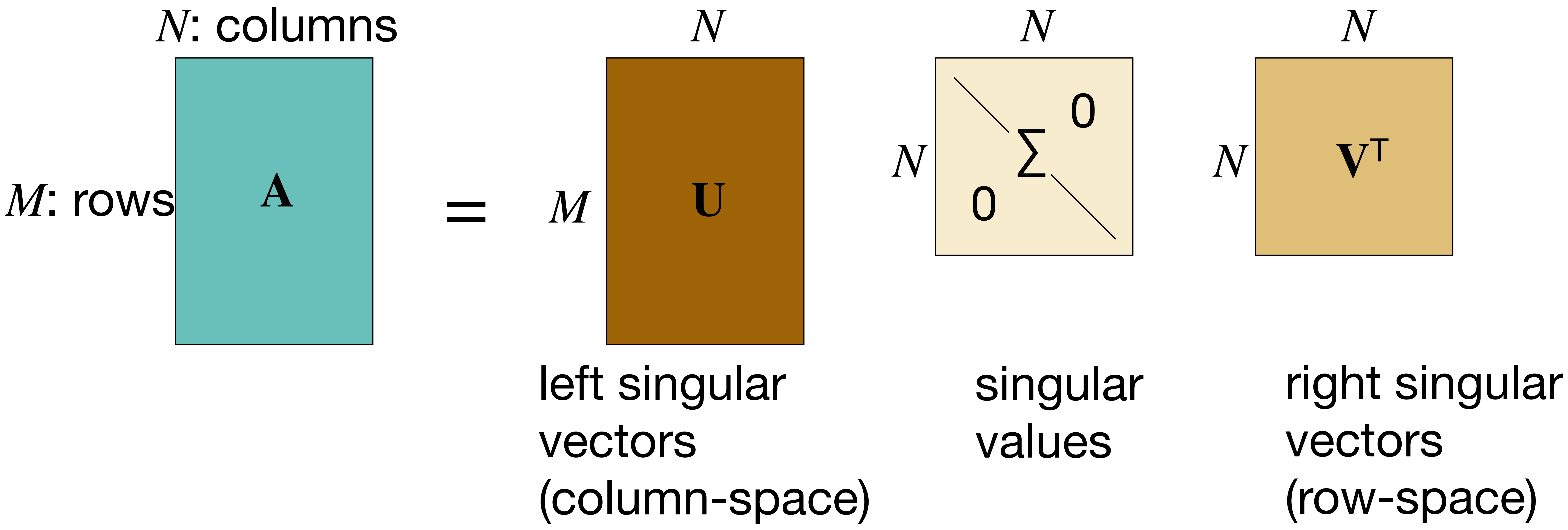
Data Reduction and Approximation

- A fundamental concept of data reduction is to remove redundant and irrelevant information while preserving the relevant features
  - e.g. through frequency analysis by projection onto pre-defined bases, or extraction of data intrinsic principal components
    - identify spatio-temporal and frequency redundancies
  - maintain strongest and most significant signal components
- Data reduction linked to concepts and techniques of data compression, noise reduction as well as feature extraction and recognition/extraction

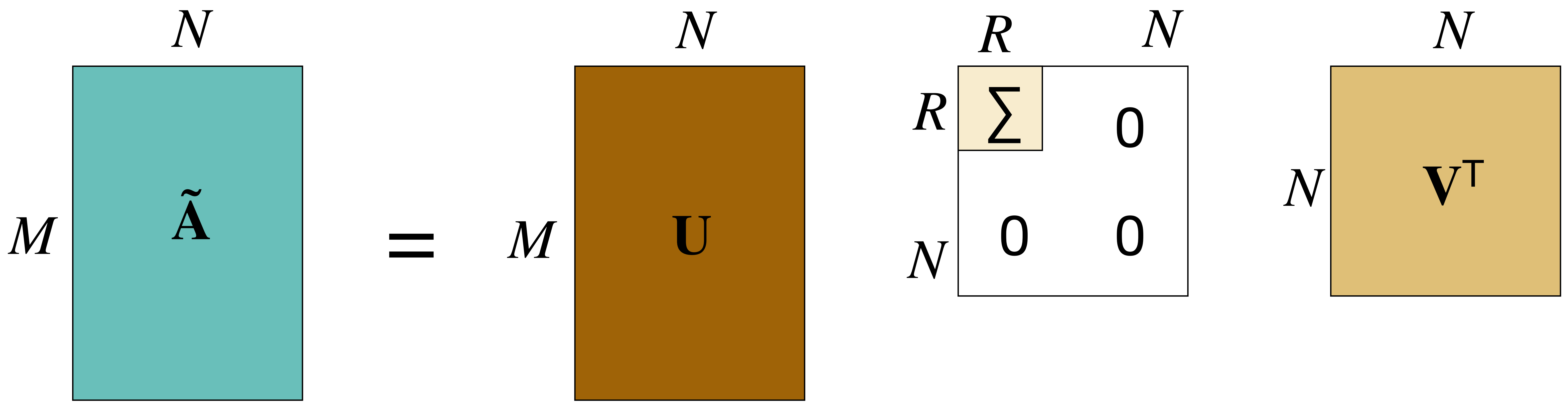
Data Approximation using SVD

- Singular Value Decomposition (SVD) standard tool for matrices, i.e., 2D input datasets
  - see also principal component analysis (PCA)



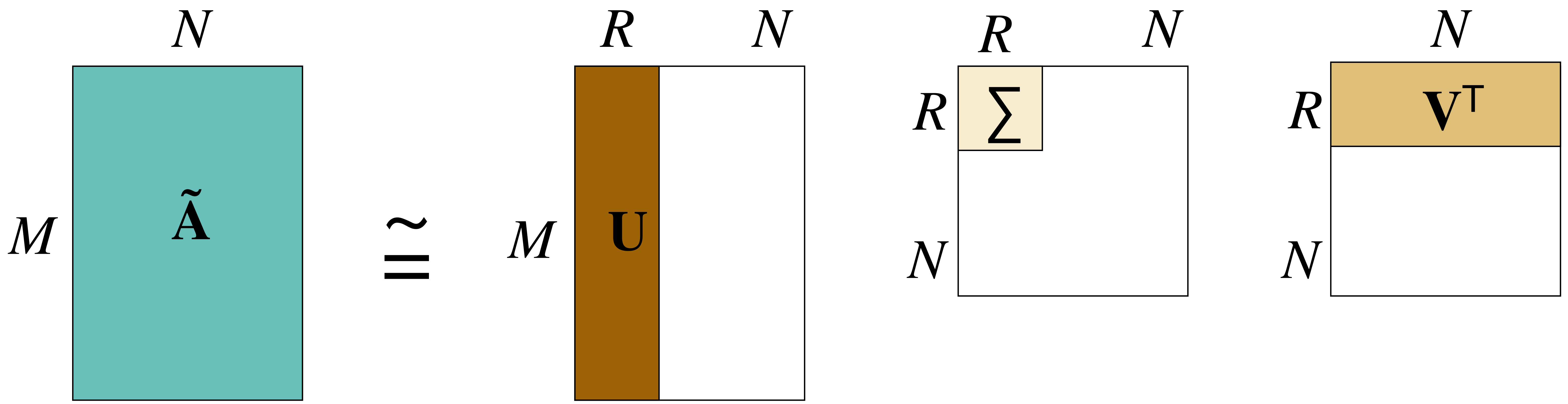
Low-rank Approximation

- Exploit ordered singular values: $s_1 \geq s_2 \geq \ldots \geq s_N$
- Select first r singular values (rank reduction)



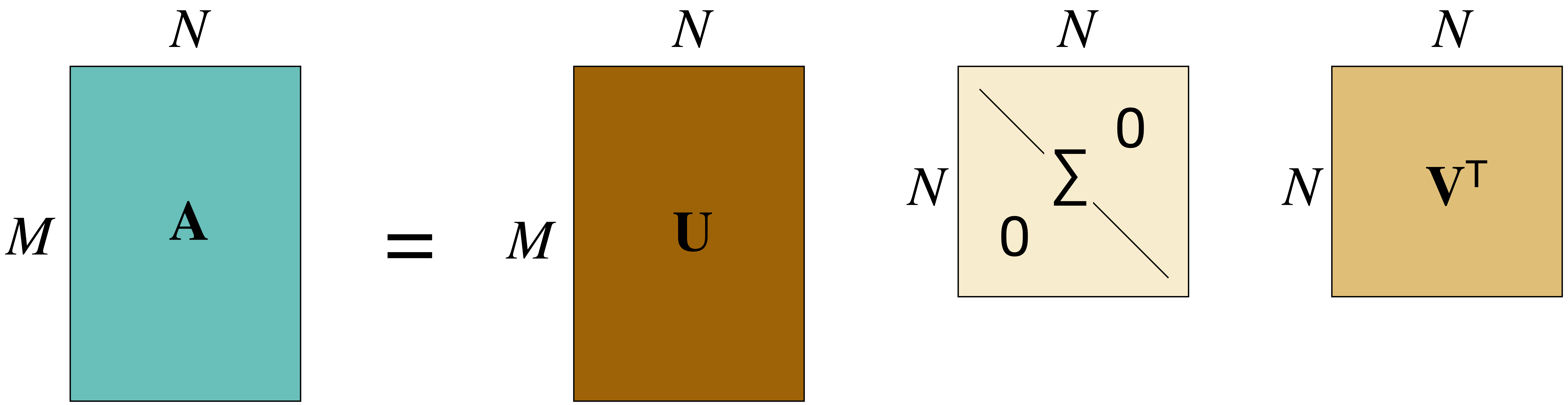
Low-rank Approximation

- Exploit ordered singular values: $s_1 \geq s_2 \geq \ldots \geq s_N$
- Select first r singular values (rank reduction)
  - use only bases (singular vectors) of corresponding subspace

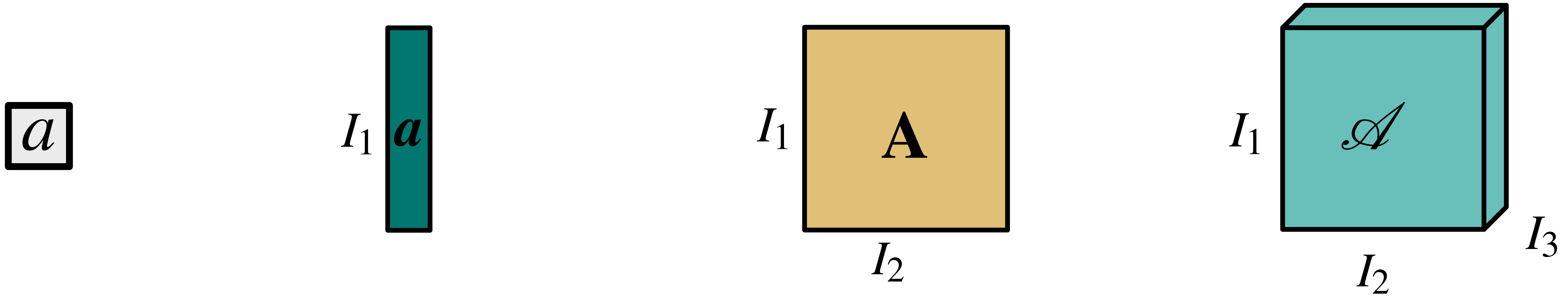


Matrix SVD Properties

- Matrix SVD
  - rank reducibility
  - orthonormal row/column matrices



What is a Tensor?



$$\mathscr{A}^{N-\text{th order}} \in \mathbb{R}^{I_1 \times I_2 \cdots \times I_N}$$

$i_1 = 1, \ldots, I_1$ | $i_2 = 1, \ldots, I_2$ | $i_3 = 1, \ldots, I_3$

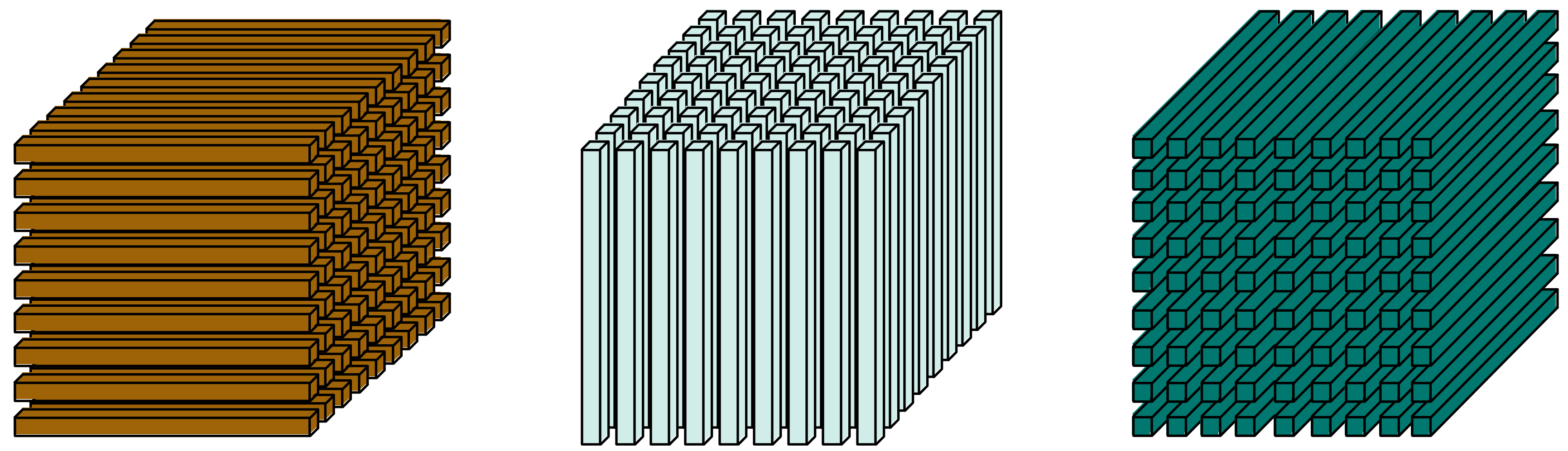
0-order tensor | 1st-order tensor | 2nd-order tensor | 3rd-order tensor | ...

- Data sets are often multidimensional arrays (tensors)
  - images, image collections, video, volume data etc.

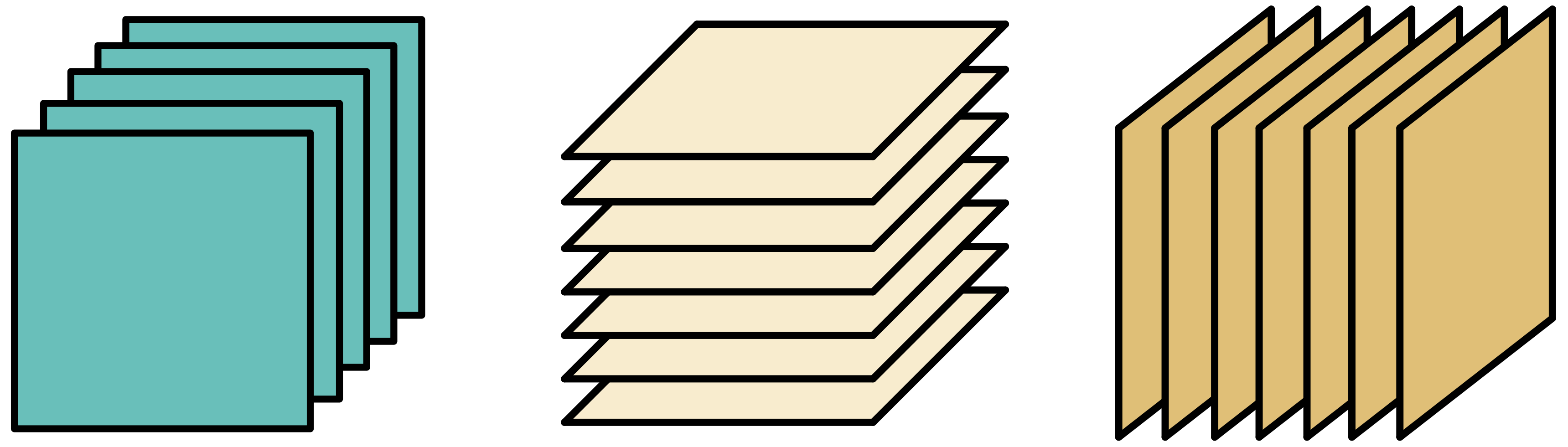
Fibers and Slices

- Individual elements of a vector $\boldsymbol{a}$ are given by a_{i1}, from a matrix $\mathbf{A}$ by $a_{i1,i2}$ and from a tensor $\mathscr{A}$ by $a_{i1,i2,i3}$

- The generalization of rows, columns (and tubes) is a *fiber* in a particular mode

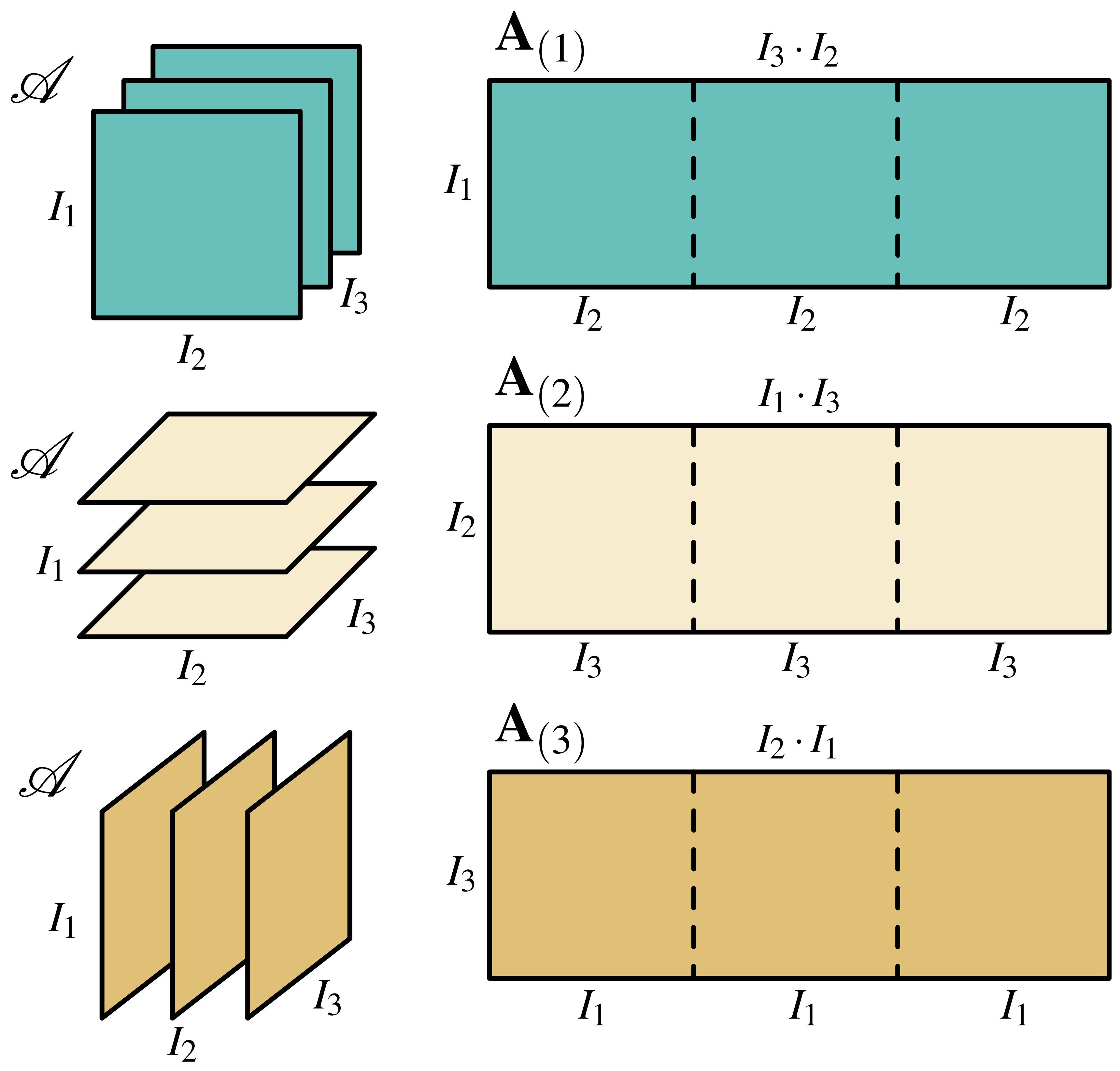


- Two dimensional sections of a tensor are called slices
  - frontal, horizontal and lateral for $\mathscr{A} \in \mathbb{R}^3$



Unfolding and Ranks

- Operations with tensors often performed as matrix operations using unfolded tensor representations
  - different tensor unfolding strategies possible
- Forward cyclic unfolding $\mathbf{A}_{(n)}$ of a 3rd order tensor $\mathscr{A}$ (or 3D volume)
- The n-rank of a tensor is typically defined on an unfolding
  - n-rank $R_n = \mathrm{rank}_n(\mathscr{A}) \; = \mathrm{rank}(\mathbf{A}_{(n)})$
  - multilinear rank-$(R_1, R_2, \ldots, R_N)$ of $\mathscr{A}$

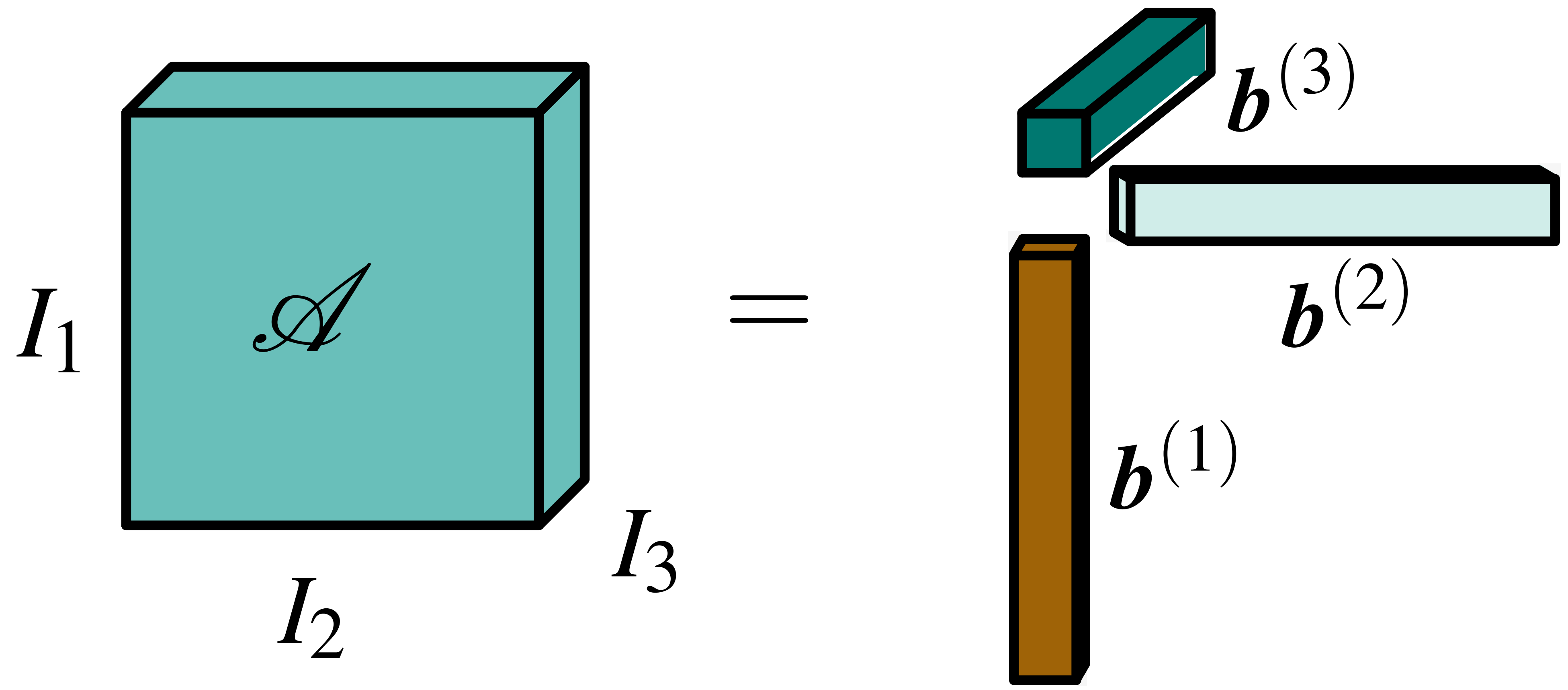


Rank-one Tensor

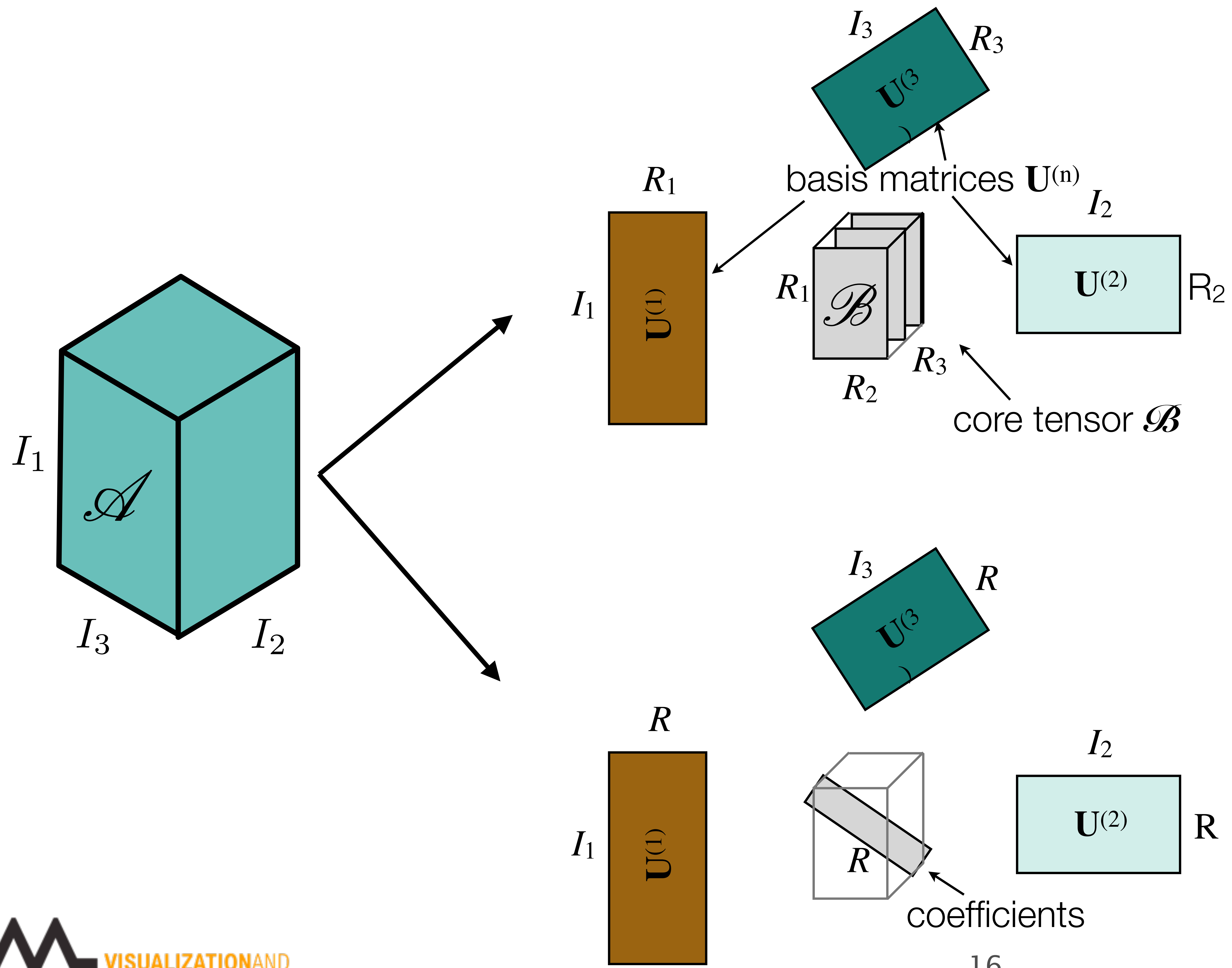
- N-mode tensor $\mathscr{A} \in \mathbb{R}^{I_1 \times \ldots \times I_N}$ that can be expressed as the outer product of N vectors
  - Kruskal tensor

$$\mathscr{A} = \boldsymbol{b}^{(1)} \circ \boldsymbol{b}^{(2)} \circ \ldots \circ \boldsymbol{b}^{(N)}$$

- Useful to understand principles of rank-reduced tensor reconstruction
  - linear combination of rank-one tensors



Tensor Decomposition Models



Tucker

- Three-mode factor analysis (**3MFA/Tucker3**) [Tucker, 1964+1966]
- Higher-order SVD (**HOSVD**) [De Lathauwer et al., 2000a]

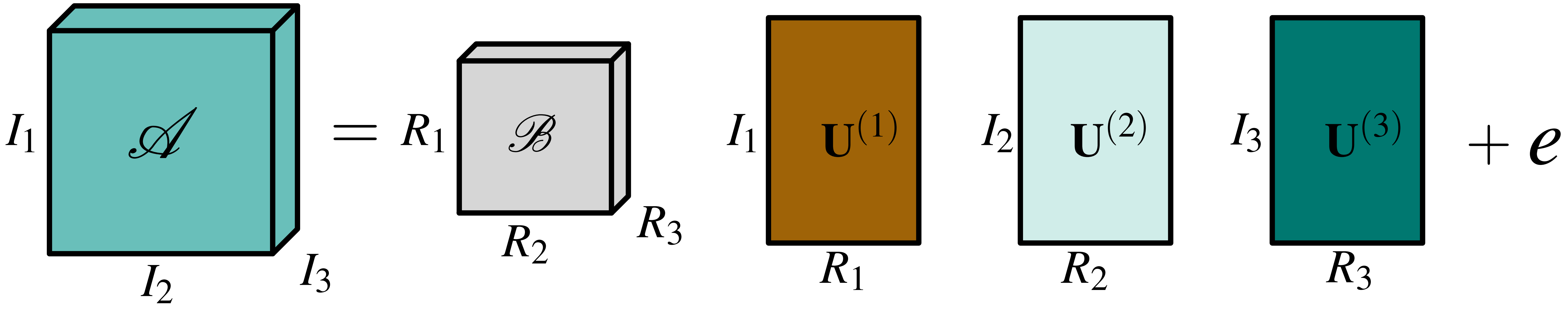
CP

- **PARAFAC** (parallel factors) [Harshman, 1970]
- **CANDECOMP** (CAND) (canonical decomposition) [Caroll & Chang, 1970]

Tucker Model

- Higher order tensor $\mathscr{A} \in \mathbb{R}^{I_1 \times \ldots \times I_N}$ represented as a product of a core tensor $\mathscr{B} \in \mathbb{R}^{R_1 \times \ldots \times R_N}$ and N factor matrices $\mathbf{U}^{(n)} \in \mathbb{R}^{I_n \times R_n}$
  - using n-mode products $\times_n$

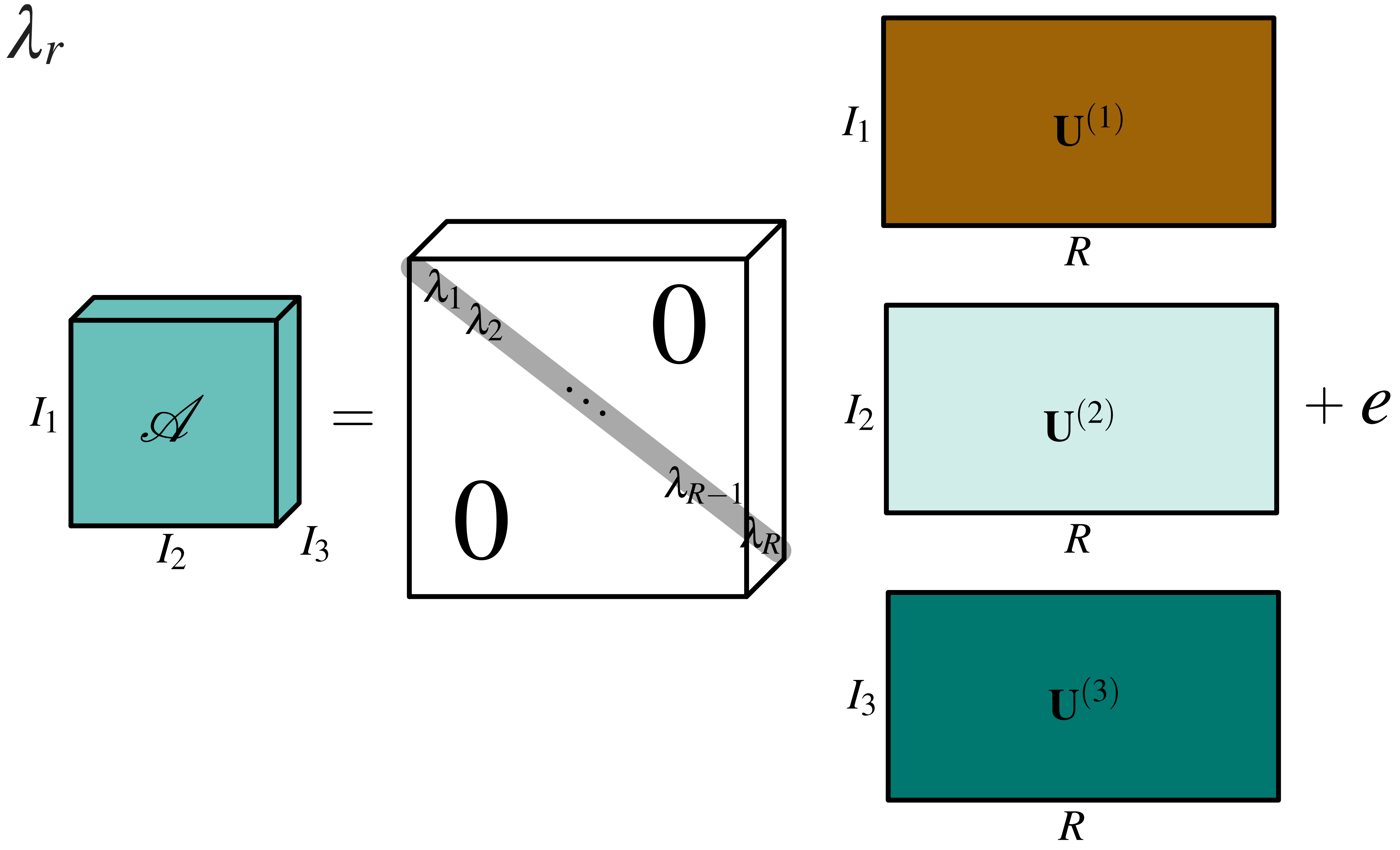
$$\mathscr{A} = \mathscr{B} \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \times_3 \cdots \times_N \mathbf{U}^{(N)} + \varepsilon$$



CANDECOMP-PARAFAC Model

- *Canonical decomposition* or *parallel factor analysis* model (CP)
- Higher order tensor $\mathscr{A}$ factorized into a sum of rank-one tensors
  - normalized column vectors $\boldsymbol{u}_r^{(n)}$ define factor matrices $\mathbf{U}^{(n)} \in \mathbb{R}^{I_n \times R}$ and weighting factors λ_r

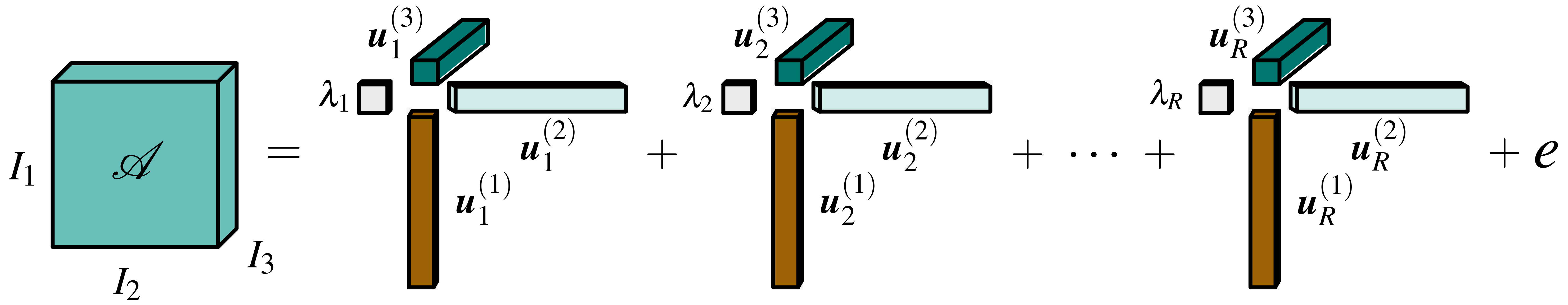
$$\mathscr{A} = \sum_{r=1}^{R} \lambda_r \cdot \boldsymbol{u}_r^{(1)} \circ \boldsymbol{u}_r^{(2)} \circ \ldots \boldsymbol{u}_r^{(N)} + \varepsilon$$



Linear Combination of Rank-one Tensors

- The CP model is defined as a linear combination of rank-one tensors

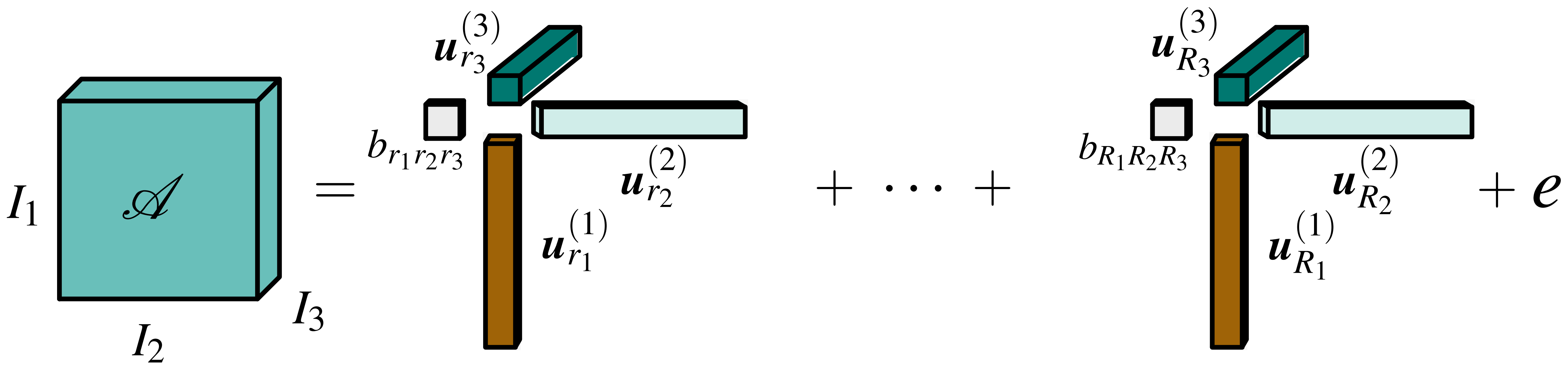
$$\mathscr{A} = \sum_{r=1}^{R} \lambda_r \cdot \boldsymbol{u}_r^{(1)} \circ \boldsymbol{u}_r^{(2)} \circ \ldots \boldsymbol{u}_r^{(N)} + \varepsilon$$



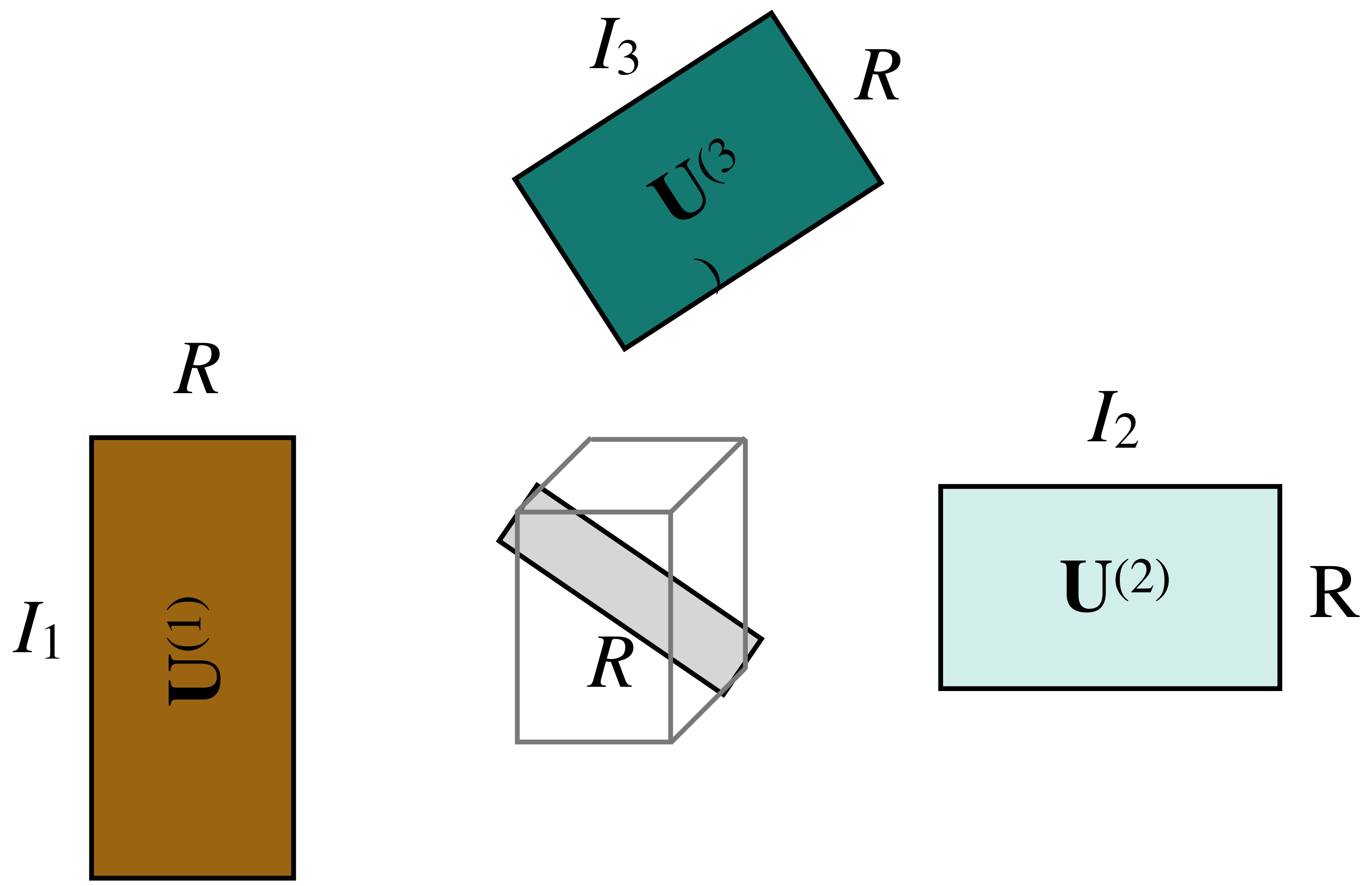
Linear Combination of Rank-one Tensors

- The CP model is defined as a linear combination of rank-one tensors
- The Tucker model can be interpreted as linear combination of rank-one tensors

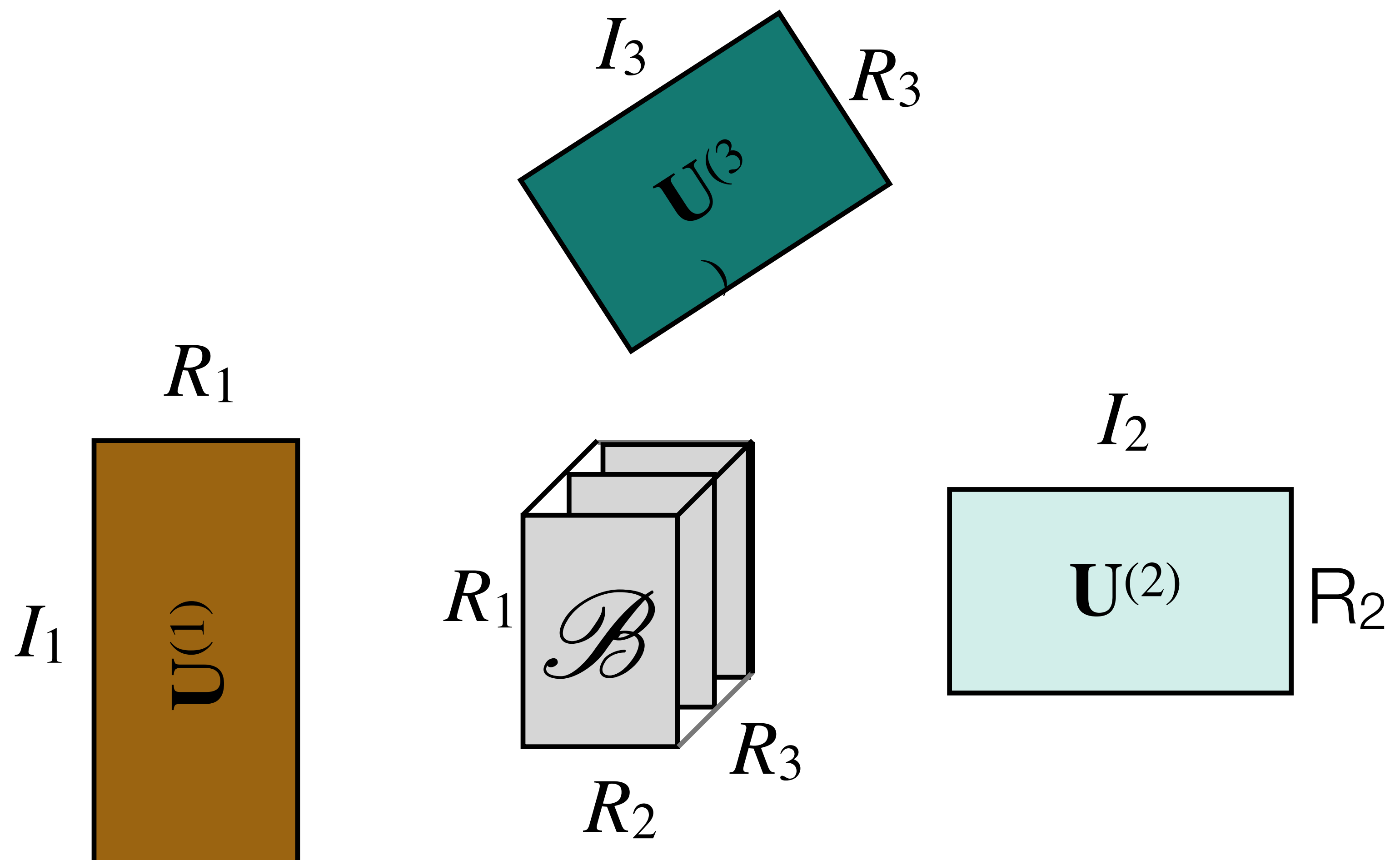
$$\mathscr{A} = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \cdots \sum_{r_N=1}^{R_N} b_{r_1 r_2 \ldots r_N} \cdot \boldsymbol{u}_{r_1}^{(1)} \circ \boldsymbol{u}_{r_2}^{(2)} \circ \ldots \boldsymbol{u}_{r_N}^{(N)} + \varepsilon$$



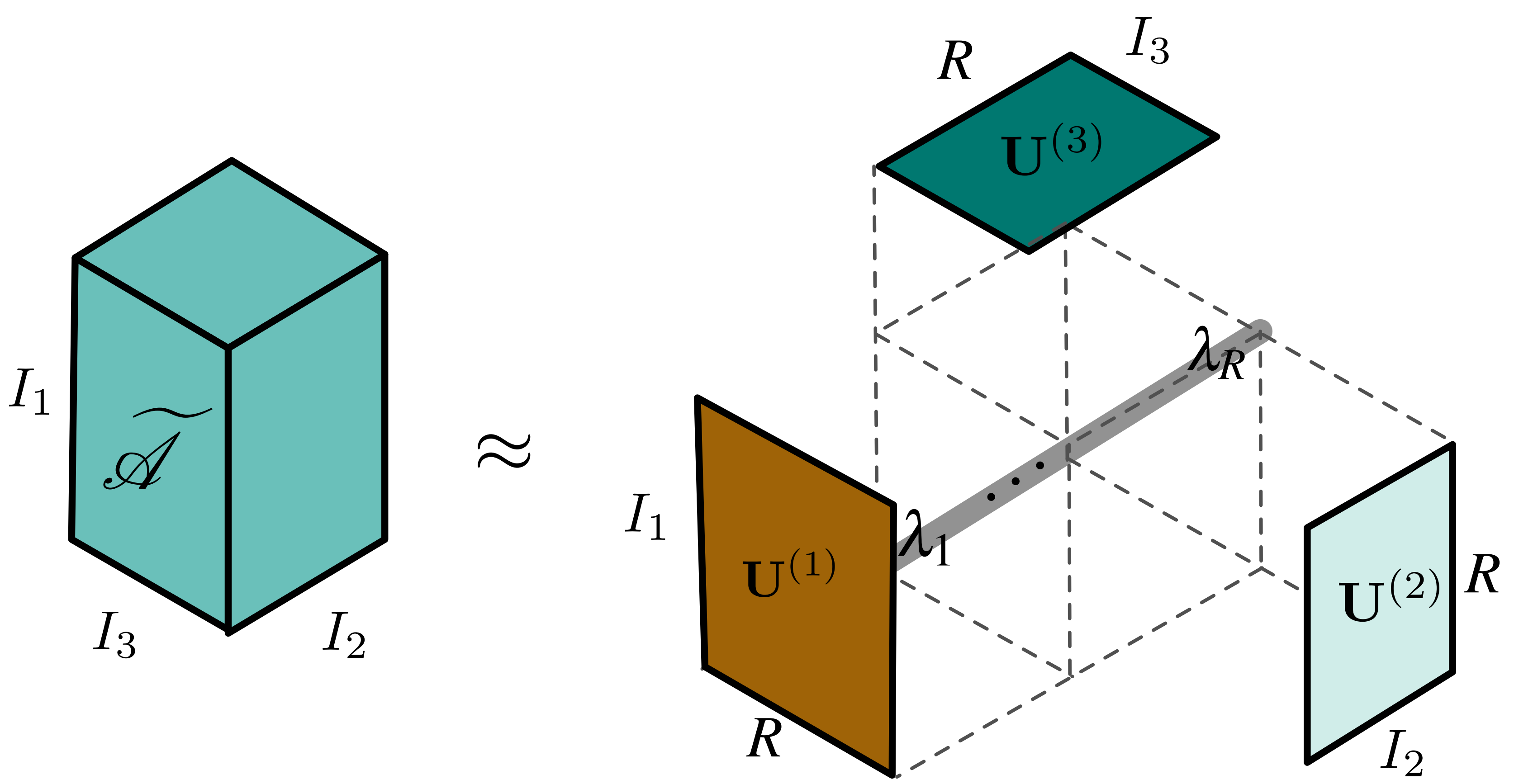
CP a Special Case of Tucker

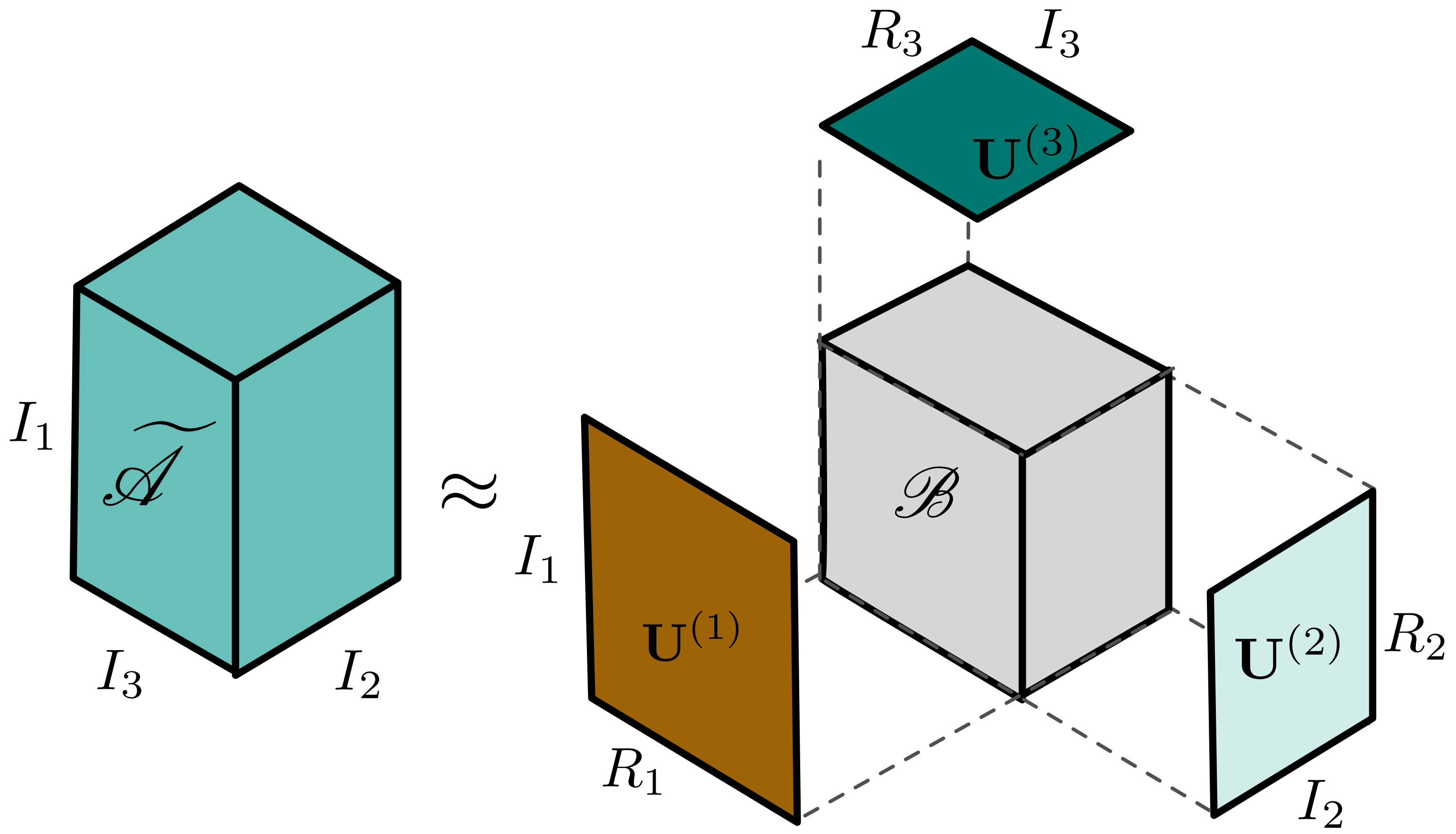


CP



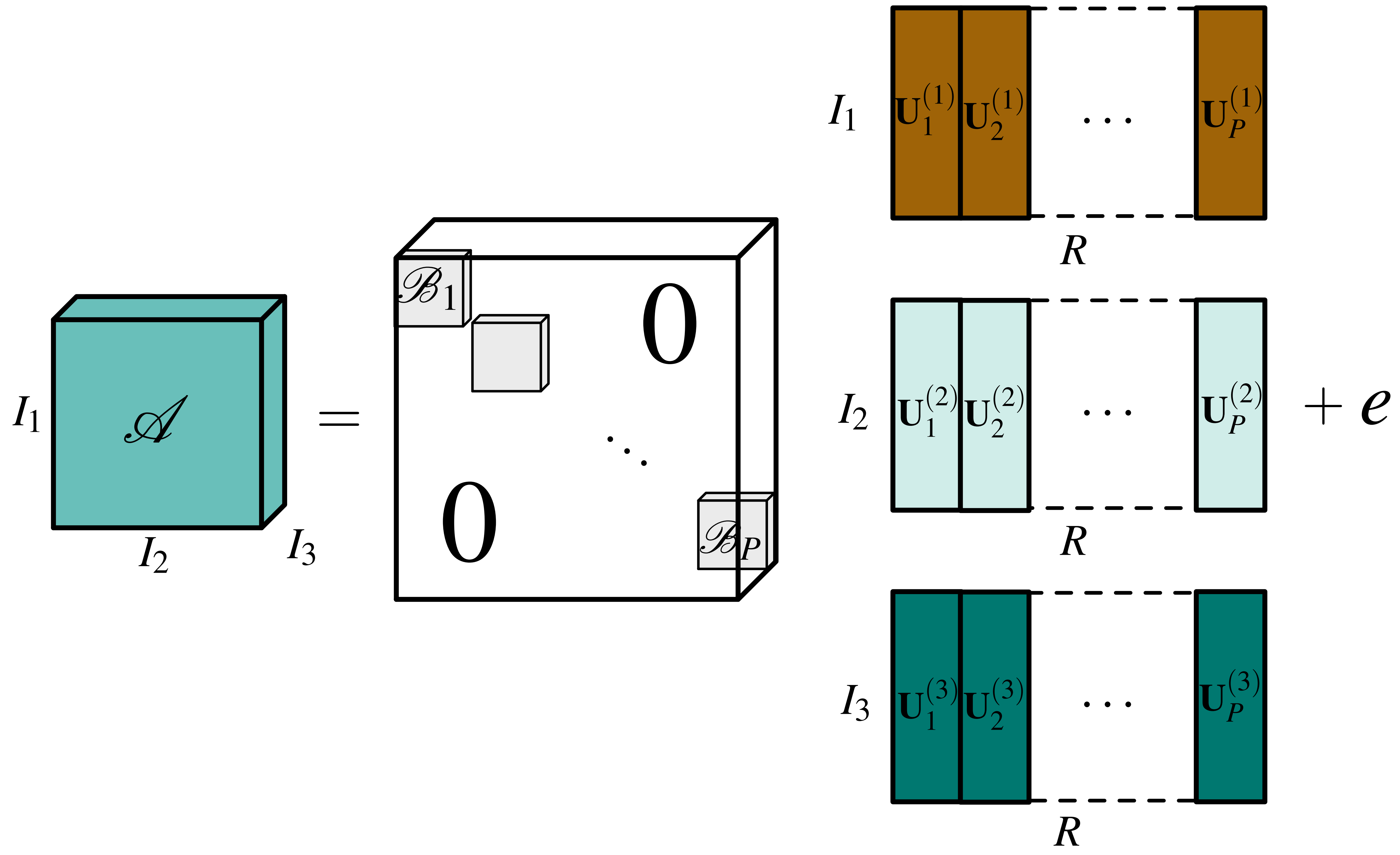
Tucker





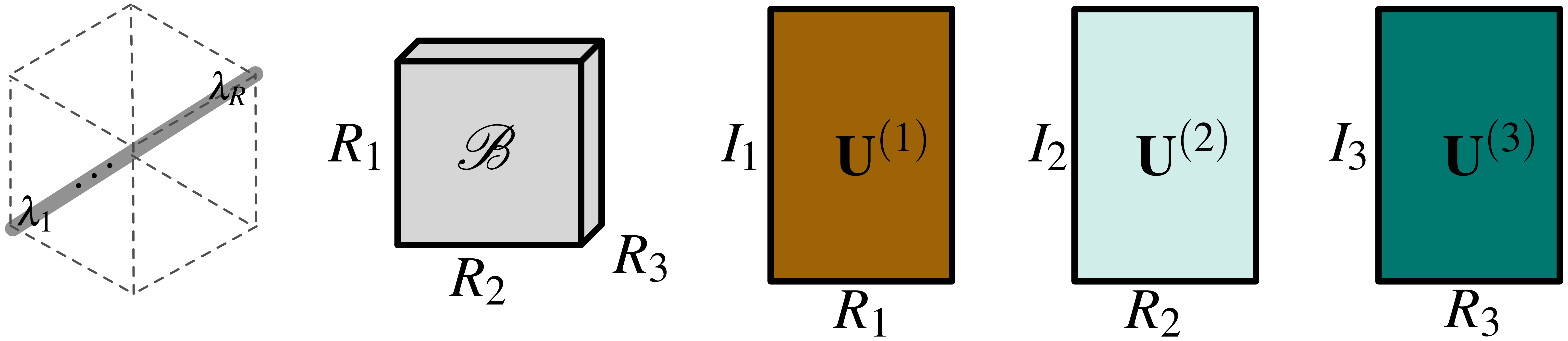
Generalizations

- Any special form of core and corresponding factor matrices
  - e.g. blocks along diagonal



Reduced Rank Approximation

- Full reconstruction using a Tucker or CP model may require excessively many coefficients and wide factor matrices
  - large rank values R (CP), or $R_1, R_2 \ldots R_N$ (Tucker)
- Quality of approximation increases with the rank, and number of column vectors of the factor matrices
  - best possible fit of these bases matrices discussed later



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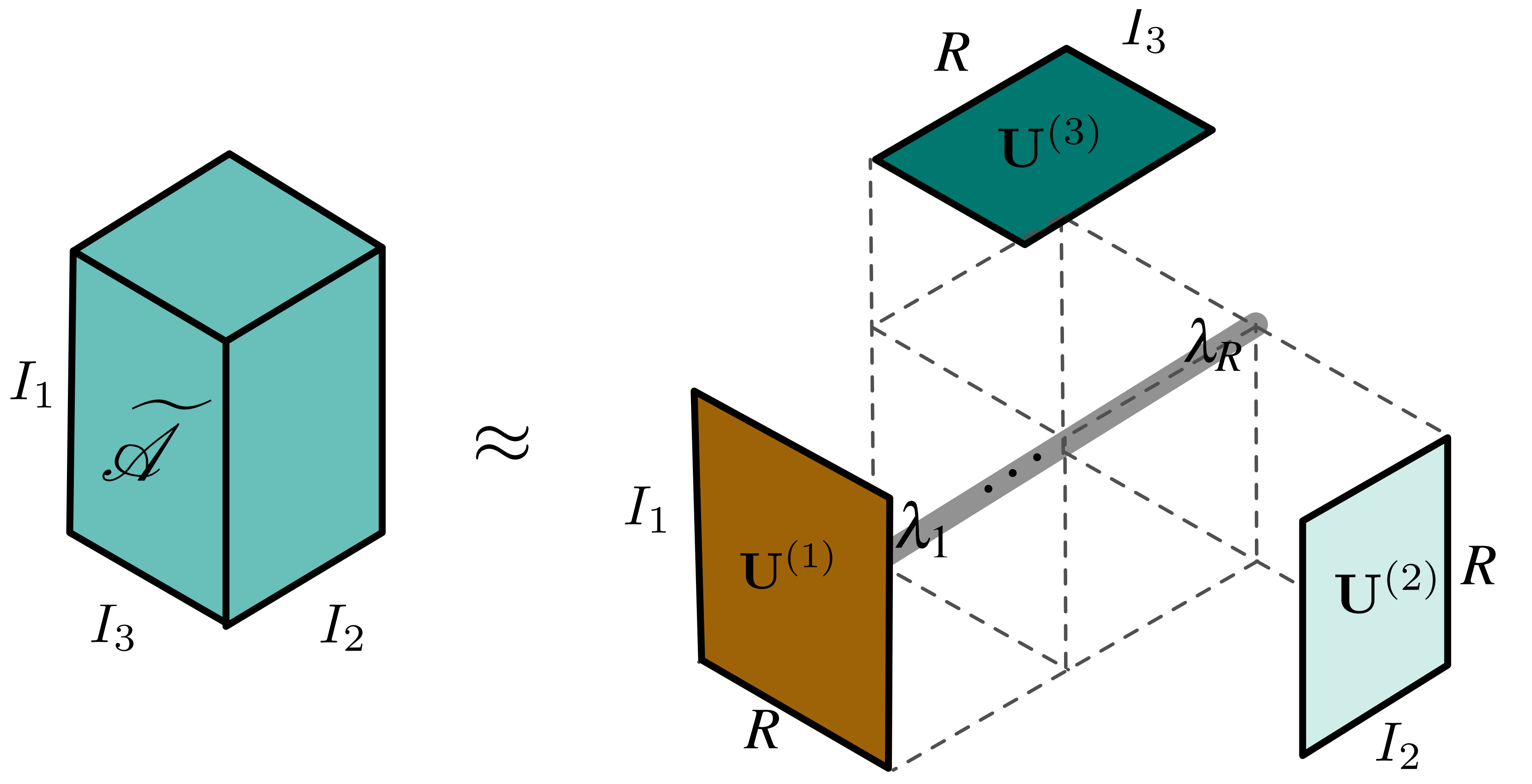
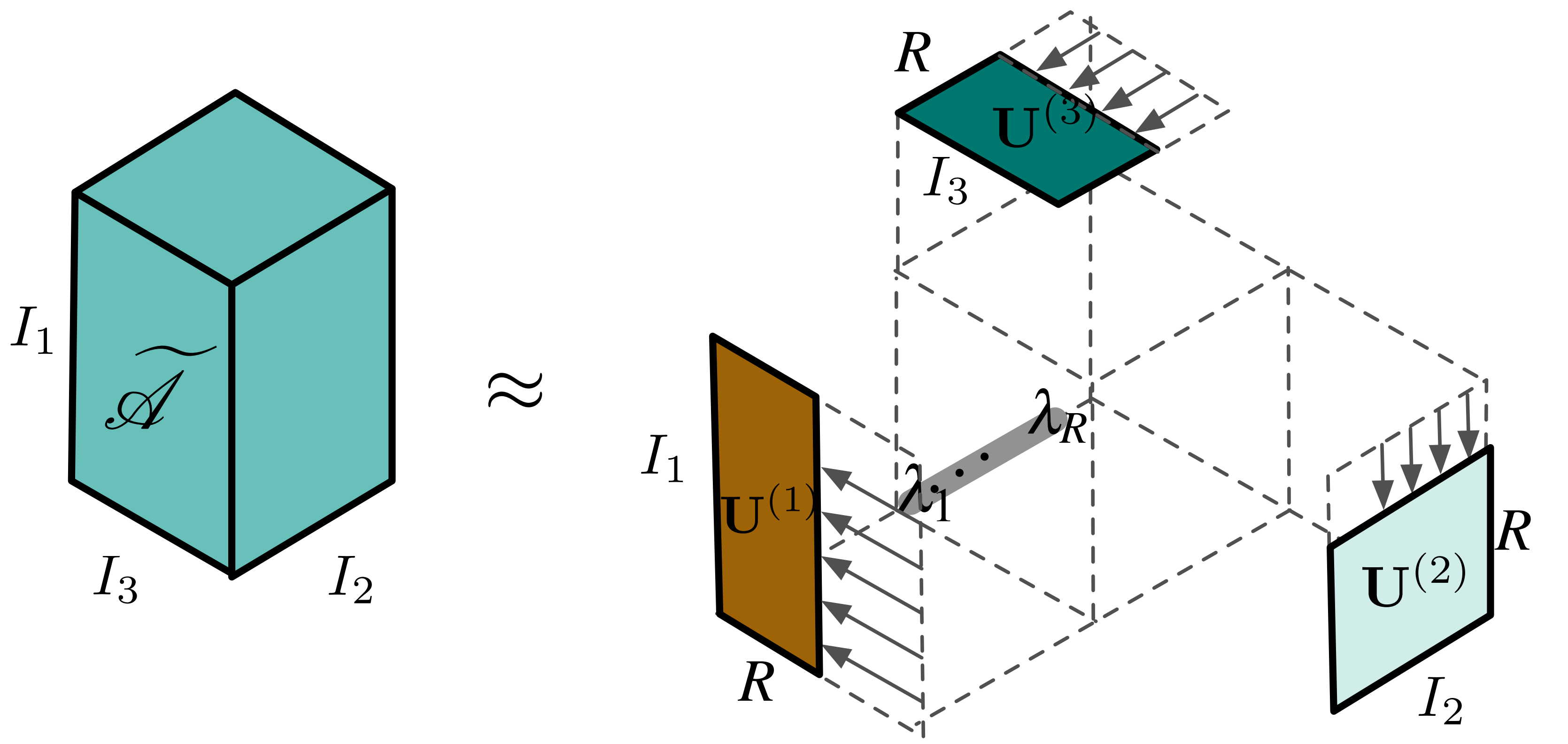
Rank-R Approximation

- Approximation of a tensor as a linear combination of ranke-one tensors using a limited number R of terms
  - CP model of limited rank R

$$\widetilde{\mathscr{A}} = \sum_{r=1}^{R} \lambda_r \cdot \boldsymbol{u}_r^{(1)} \circ \boldsymbol{u}_r^{(2)} \circ \ldots \boldsymbol{u}_r^{(N)}$$

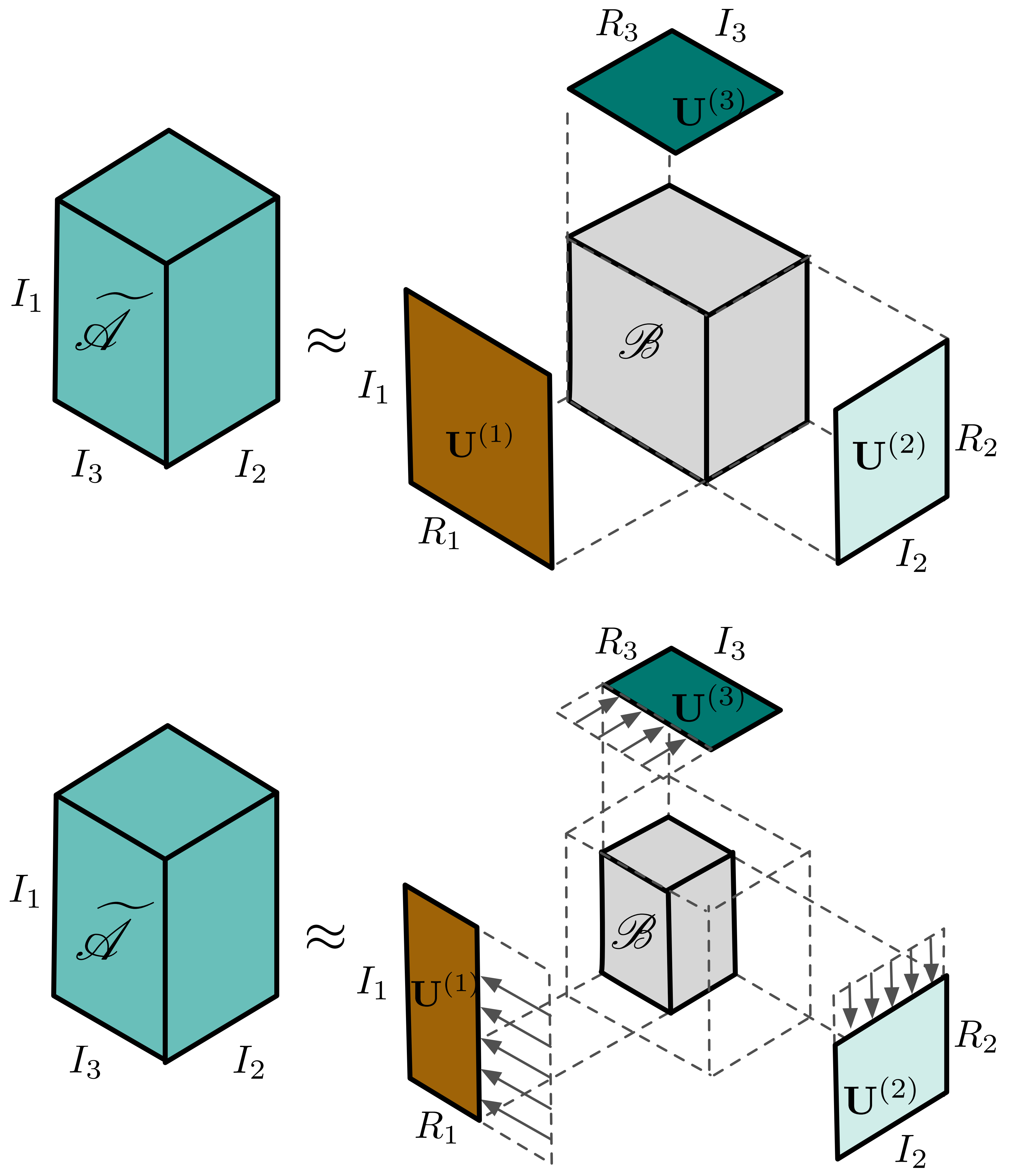




Rank-$(R_1, R_2, \ldots, R_N)$ Approximation

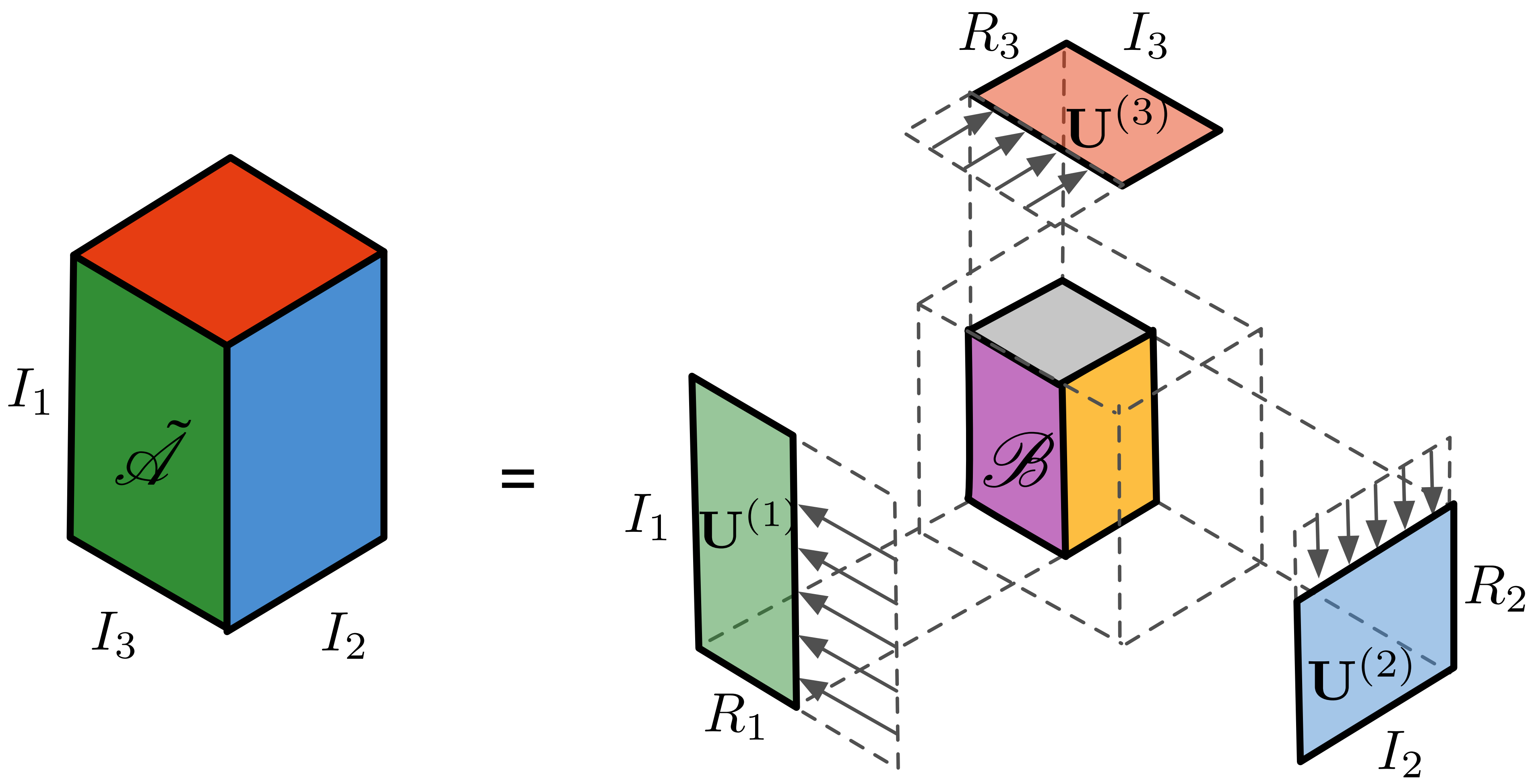
- Decomposition into a tensor with reduced, lower multilinear $\operatorname{rank}(R_1, R_2, \ldots, R_N)$
  - $\operatorname{rank}_n(\widetilde{\mathscr{A}}) = R_n \leq \operatorname{rank}_n(\mathscr{A}) = \operatorname{rank}(\mathbf{A}_{(n)})$
- ***n***-mode products of factor matrices and core tensor in a given reduced rank space
  - Tucker model with limited ranks R_i

$$\widetilde{\mathscr{A}} = \mathscr{B} \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \times_3 \cdots \times_N \mathbf{U}^{(N)}$$



Best Rank Approximation

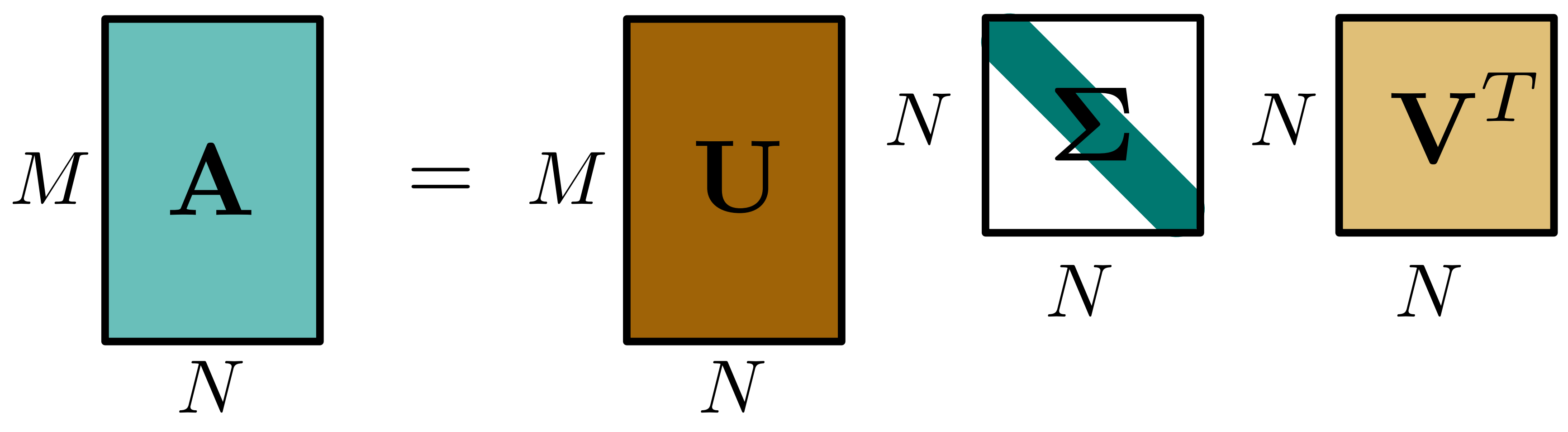
- Rank reduced approximation that minimizes least-squares cost
  - $\widetilde{\mathscr{A}} = \arg\min(\widetilde{\mathscr{A}}) \left\| \mathscr{A} - \widetilde{\mathscr{A}} \right\|^2$
- Alternating least squares (ALS) iterative algorithm that converges to a minimum approximation error based on the Frobenius norm $\|\ldots\|_F$
  - rotation of components in basis matrices



$$\widetilde{\mathscr{A}} = \mathscr{B} \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \times_3 \mathbf{U}^{(3)}$$

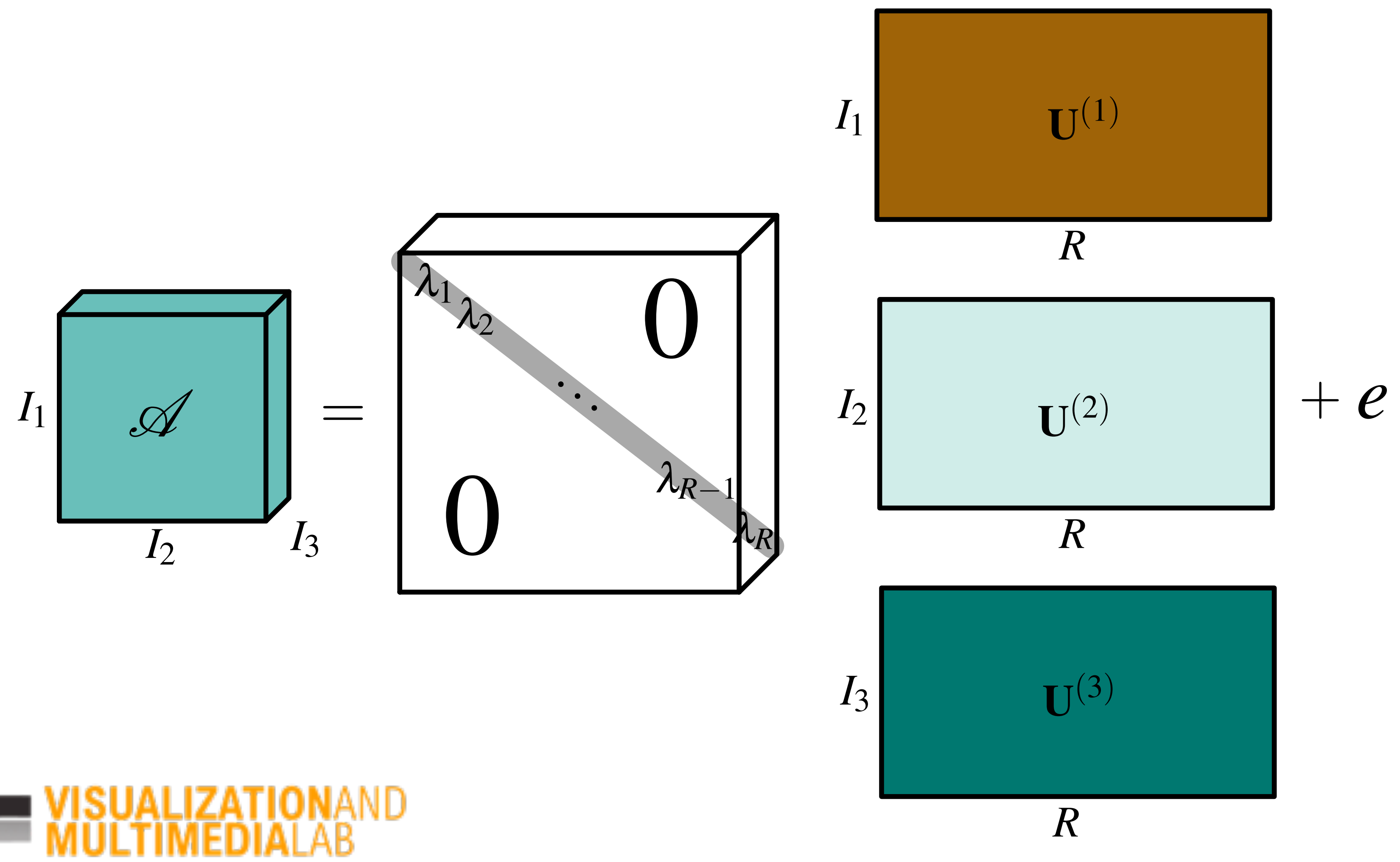
typical high-quality data reduction:
$R_k \leq I_k / 2$

Generalization of the Matrix SVD



higher orders

CP model



Tucker model